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A non-linear wear-contact problem for a Winkler foundation with an increasing contact area☆

I.A. Soldatenkov

Moscow, Russia

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ABSTRACT

A wear-contact problem for a Winkler foundation is considered in the case when the rate of wear depends nonlinearly on the contact pressure and the contact area increases. The corresponding integral and differential equations are obtained. A successive approximation procedure is proposed which enables an exact solution of the problem in the space of continuous functions to be found. The property of non-negativity of the contact pressure when it has non-negative initial values is established. It is shown, using a qualitative analysis and calculations, that the non-linearity of the wear law can have a considerable effect on the behaviour of the contact pressure during the wear process.

The model of a Winkler foundation, which establishes a linear algebraic relationship between the normal displacement of the surface of a body and the contact pressure, is widely used in engineering practice to calculate the bending of different structural components (beams and plates) on a deformable foundation.[1] The Winkler model can also be used to describe the deformation of different kinds of surface structures which are encountered at friction joints. A thin elastic layer[2] and the surface roughness of bodies[3,4] are examples of such structures and, if the main body (the underlying material) is sufficiently rigid, such that its deformation can be neglected, the Winkler model will determine the dependence of the deformation displacement of the surface of the body on the contact pressure. The simplicity of this model enables us to obtain exact solutions of the corresponding wear-contest problems.

1. Formulation of the problem

We consider a deformable foundation with a plane surface. With it, we associate a system of coordinates $Oxyz$, in which the x and z axes coincide with the surface of the base, while the y axis is directed along the outward normal to it (Fig. 1). We will assume that an absolutely rigid punch, the generatrix of the surface of which is parallel to the z axis, is indented into the foundation translationally along the y axis. We shall represent the contact area between the punch and the foundation by the segment $[-a, b]$ of the x axis, where $a>0$, $b>0$. Simultaneously, the punch slides over the foundation along the z axis at a velocity υ which is constant in magnitude, as a result of which the wear of the punch occurs. We shall denote the corresponding change in the quantities with time by the argument t, taking the instant $t=0$ as the start of the wear. The initial form of the punch is specified by a smooth, monotonically decreasing (increasing) function $g(x)$, $g(0)=0$ when $x<0$, $(x>0)$. We will assume that, in the case of an unchanged sliding velocity υ, the rate of linear wear W of the punch is determined by the value $p=-\sigma_y|_{y=0}$ of the contact pressure according to the wear law

$$\partial W(x,t)/\partial t = F(p(x,t)), \quad W(x,0) = 0 \tag{1.1}$$

where $F(p)$ is a known function for which it is assumed that

$$F(p) = 0, \quad p \le 0; \quad F(p) > 0, \quad p > 0 \tag{1.2}$$

$$0 \le F''(p) \in C(-\infty,\infty) \tag{1.3}$$

and that $F(p)$ satisfies the lipschits condition

$$|F(p_2) - F(p_1)| \le F_{1M}|p_2 - p_1|, \quad p_{1,2} \in (-\infty,\infty) \tag{1.4}$$

where F_{1M} is a known constant. It is obvious that, in the case of condition (1.2), the function $W(x,t)$ satisfying equalities (1.1) is non-negative.

Suppose that, due to wear of the punch, the size of the contact area increases smoothly over a certain time interval $[0, t_*]$, $t_*>0$:

$$0 < a'(t) \in C[0,t_*], \quad 0 < b'(t) \in C[0,t_*] \tag{1.5}$$

which enables us to determine the rate of increase in a: $V(t) \equiv a'(t)$. Assumptions (1.5) also ensure the existence of a monotonically increasing function $t(a)$ which is the inverse of $a(t)$. This enables us to use the dimension a of the contact area as the time parameter

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E-mail address: isoldat@mail.ru.

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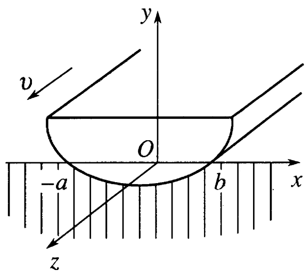

Fig. 1.

instead of t so that

$$b(a) \equiv b(t(a)), \quad V(a) \equiv V(t(a)), \quad p(x, a) \equiv p(x, t(a)),$$
$$W(x, a) \equiv W(x, t(a))$$

The argument a of the function $b(a)$ will henceforth be omitted wherever it cannot give rise to any misunderstanding.

Well-known inverse function theorems[5] with assumptions (1.5) enable us to establish the following properties of the functions $t(a)$ and $V(a)$

$$t(a) \in C^1[a_0, a_*], \quad t'(a) = 1/V(t) \equiv 1/V(a) \tag{1.6}$$

$$0 < V(a) \in C[a_0, a_*] \tag{1.7}$$

Here and henceforth, $a_0 = a(0)$, $b_0 = b(0)$ and $a_* = a(t_*)$, $b_* = b(t_*)$.

Changing from the variable t to the variable a, the form

$$\partial W(x, a)/\partial a = F(p(x, a))/V(a), \quad W(x, a_0) = 0 \tag{1.8}$$

can be attributed to the wear law (1.1) using the rule for the differentiation of a complex function.

We denote the vertical displacement (along the y axis) of the surface of the foundation as the result of its deformation by u_y so that, according to the Winkler model,

$$u_y(x, a) = -Ap(x, a)$$

Substituting this expression into the condition for the punch and the foundation to be in contact we obtain the equality

$$Ap(x, a) + W(x, a) = \delta(a) - g(x), \quad x \in [-a, b] \tag{1.9}$$

where the quantity δ defines the vertical displacement of the punch. The condition that the contact pressure at the ends of the contact area is equal to zero and the condition for the equilibrium of the punch

$$p(-a, a) = p(b, a) = 0 \tag{1.10}$$

$$\int_{-a}^{b} p(x, a)dx = Q \tag{1.11}$$

also hold, where Q is the constant vertical load per unit length of the punch along the z axis.

With the above assumptions, the dimensions a and b of the contact area and the displacement of the punch δ turn out to be related to one another by the equalities

$$g(b(a)) = g(-a) = \delta(a) \tag{1.12}$$

Actually, outside the contact area when a and b increase, there is no wear and, according to the Winkler model, there is also no deformation displacement u_y of the surface of the foundation. This enables us to write the relation

$$g(x) = \delta(a) + d(x, a), x \,\bar{\in}\, [-a, b] \text{ when } x \,\bar{\in}\, [-a, b]$$

in which d is the gap between the surfaces of the punch and the foundation. By virtue of the continuity of the function $g(x)$, equalities (1.12) follow from the last relation, since $d(x, a) \to 0$ when $x \to -a - 0$ or $x \to b + 0$.

If the function $x = g_+^{-1}(y)$, which is the inverse of the function $y = g_+(x) \equiv g(x)|_{x \in [b_0, b_*]}$, is introduced into the treatment, then the first equality of (1.12) enables us to determine the relation $b(a)$ in terms of the known shape of the punch:

$$b(a) = g_+^{-1}[g(-a)], \quad 0 < b'(a) = -g'(-a)/g'(b(a)) \in C[a_0, a_*] \tag{1.13}$$

We now introduce the set Π_* of points x, a into the treatment. The abscissa x of these points lies within the contact area $[-a, b(a)]$ for any $a \in [a_0, a_*]$. Relations (1.13) enable us to write

$$\Pi_* \equiv \{x, a\colon x \in [-a, b(a)], a \in [a_0, a_*]\} = \{x, a\colon x \in [-a_*, b_*],$$
$$a \in [\psi(x), a_*]\} \tag{1.14}$$

where the function $\psi(x)$ determines the distance from the x-axis to the lower boundary of the set Π_* and has the form

$$\psi(x) = \begin{cases} -x, & x \in [-a_*, -a_0] \\ a_0, & x \in (-a_0, b_0) \\ b^{-1}(x), & x \in [b_0, b_*] \end{cases}$$

The function $a = b^{-1}(x)$ is the inverse of the known function $x = b(a)$ of the form (1.13). By virtue of relations (1.13), $\psi(x) \in C[-a_*, b_*]$.

It is required to find the functions $p(x, a)$, $W(x, a)$, $\delta(a)$, $V(a)$ which satisfy Eqs. (1.8)–(1.11) and relation (1.7) when $x, a \in \Pi_*$.

2. Solution of the problem

We shall seek the contact pressure $p(x, a)$ in the space of functions which are continuous in the set Π_*:

$$p(x, a) \in C(\Pi_*) \tag{2.1}$$

We define the desired rate of increase of a of the contact area by the expression

$$V(a) = (\mathfrak{R}p)(a) \equiv -\frac{1}{2l(a)g'(-a)} \int_{-a}^{b} F(p(x, a))dx,$$
$$a \in [a_0, a_*]; \quad l(a) = \frac{a + b(a)}{2} \quad (2.2) \tag{2.2}$$

Expression (2.2) is formally obtained if the contact condition (1.9) is initially integrated over $x \in [-a, b]$ and the result is then differentiated with respect to a while, at the same time, taking account of the wear law (1.8) and the equilibrium condition (1.11). The following lemma holds for the integral on the right-hand side of equality (2.2).

Lemma 1. *Suppose M is the set of bounded functions $\varphi(x)$ which are continuous almost everywhere in the interval $[-a, b]$ and satisfy the condition*

$$\int_{-a}^{b} \varphi(x)dx = Q \tag{2.3}$$

where Q is a certain constant, and suppose the function $F(S)$ is given such that $F''(S) \in C'(-\infty, \infty)$.

Then,

$$\min_{\varphi(x)\in M}\int_{-a}^{b}F(\varphi(x))dx = (a+b)F(\bar\varphi) \text{ when } F''(s)\ge 0$$

$$\max_{\varphi(x)\in M}\int_{-a}^{b}F(\varphi(x))dx = (a+b)F(\bar\varphi) \text{ when } F''(s)\le 0$$

where $S\in(-\infty,\ \infty)$ *and* $\bar\varphi = Q/(a+b)$ *is the mean value of the function* $\varphi(x)\in M$.

Proof. Using Taylor's formula, we write the equality

$$F(\varphi(x)) = F(\bar\varphi+\rho(x)) = F(\bar\varphi)+F'(\bar\varphi)\rho(x)+R(x);\quad \rho(x) = \varphi(x)-\bar\varphi \tag{2.4}$$

in which $R(x)=F'(\varphi_1(x))\rho^2(x)/2$, where the values of $\varphi_1(x)$ lie between $\bar\varphi$ and $\varphi(x)=\bar\varphi+\rho(x)$. Integration of equality (2.4) over the $[-a, b]$ completes the proof.

The lemma which has been proved enables us to formulate the following assertion.

Assertion 1. If the functions $g(x)$ and $F(p)$ satisfy the conditions stipulated above and $p(x,a)$ satisfies the equilibrium condition (1.11) and possesses the continuity property (2.1), then relation (1.7) is satisfied in the case of a function $V(a)$ of the form (2.2), that is,

$$0<V_m\le V(a) = (\mathcal{R}p)(a)\in C[a_0, a_*],\quad a\in[a_0, a_*]) \tag{2.5}$$

where

$$V_m = \min_{a\in[a_0, a_*]}\left[-\frac{1}{g'(-a)}F\left(\frac{Q}{2l(a)}\right)\right]>0$$

(V_m is a known quantity).

Proof. The integral on the right-hand side of equality (2.2) is continuous over $a\in[a_0, a_*]$, which is ensured by the continuity of the functions $F(p)$, $b(a)$ (see properties (1.3), (1.13)) and condition (2.1).[5] With the constraints imposed on the function $g(x)$, this enables us to establish the continuity with respect to $a\in[a_0, a_*]$ of the whole of expression (2.2) for Va. The existence and positiveness of the quantity V_m results from properties (1.2) and (1.13) of the functions $F(p)$, $b(a)$ and, also, from the inequality $g(-a)<0$. The right-hand inequality (2.5) can be established if account is taken of the fact that, in conditions (1.3) and (1.11), the integral on the right-hand side of equality (2.2) reaches a minimum according to Lemma 1 when $p(x, a)=Q/(2l(a))$.

Assertion 1 enables us to establish a number of relations between the required functions. First of all, by virtue of relations (2.5) and, also, properties (1.3) and (2.1), we have: $V^{-1}(a)F(p(x, a))\in C(\Pi_*)$, which enables us to integrate the wear law (1.8) with respect to a and obtain the equality[5]

$$W(x, a) = \int_{\psi(x)}^{a}F(p(x, s))\frac{ds}{V(s)} \tag{2.6}$$

where, by virtue of the continuity of the function $\psi(x)$,

$$W(x, a)\in C(\Pi_*);\quad W(-a, a) = W(b, a) = 0 \tag{2.7}$$

Next, we integrate the contact condition (1.9) over $x\in[-a, b]$ and replace the resulting integral of the contact pressure with the load Q according to the equilibrium condition (1.11). From the equality obtained in this way, it is easy to arrive at the following expression

$$\delta(a) = \frac{1}{2l(a)}\left[\int_{-a}^{b}W(x, a)dx+AQ+S_g(a)\right];\quad S_g(a) = \int_{-a}^{b}g(x)dx \tag{2.8}$$

We finally obtain the equation for $p(x, a)$. Expression (2.8) for $\delta(a)$ is now substituted into the equality (1.9) and the contact condition is written in the form

$$Ap(x, a)+W(x, a) = \frac{1}{2l(a)}\left[\int_{-a}^{b}W(x, a)dx+AQ+S_g(a)\right]-g(x),$$
$$x\in[-a, b] \tag{2.9}$$

Eliminating $V(a)$ and $W(x, a)$ using expressions (2.2) and (2.6), the following integral equation for the contact pressure can be obtained

$$p(x, a) = (\mathcal{H}p)(x, a),\quad x, a\in\Pi_* \tag{2.10}$$

where the operator $\mathcal{H}$ has the form

$$(Hp)(x, a) = (\mathcal{K}\varphi)(x, a)+z(x, a) \tag{2.11}$$

$$(\mathcal{K}\varphi)(x, a) = \frac{1}{A}\left[\frac{1}{2l(a)}\int_{-a}^{b}dx\int_{\psi(x)}^{a}F(\varphi(x, s))\frac{ds}{(\mathcal{R}\varphi)(s)}-\int_{\psi(x)}^{a}F(\varphi(x, s))\frac{ds}{(\mathcal{R}\varphi)(s)}\right] \tag{2.12}$$

$$z(x, a) = \frac{1}{A}\left[\frac{AQ+S_g(a)}{2l(a)}-g(x)\right] \tag{2.13}$$

Note that the function $\delta(a)$ can also be eliminated from equality (1.9) using relation (1.12) to obtain the contact condition in the form

$$Ap(x, a)+W(x, a) = g(-a)-g(x),\quad x\in[-a, b] \tag{2.14}$$

The contact pressure $p(x, a)$ on the left-hand side of this equality satisfies condition (1.10) by virtue of equalities (1.12) and (2.7) whereas the contact condition as written in (2.9) ensures that the equilibrium condition (1.11) is satisfied for $p(x, a)$, as can be shown by integrating equality (2.9) over $x\in[-a, b]$.

Both ways of writing the contact condition (2.9) and (2.14) will be used below.

The well-known the properties of the continuity of integrals[5] depending on a parameter enable us to prove the following assertion.

Assertion 2. In the case of the constraints which have been imposed above on the functions $g(x)$ and $F(p)$, the operator $\mathcal{H}$ maps the space $C(\Pi_*)$ into itself:

$$(\mathcal{H}\varphi)(x, a)\in C(\Pi_*) \text{ when } \varphi(x, a)\in C(\Pi_*)$$

Before proceeding to find the solution of Eq. (2.10), we will make a number of estimates. Taking account of the non-negativity of the wear $W(x, a)$ and using the contact condition in the form of (2.14), we write

$$Ap(x, a)\le g(-a)-g(x)\le 2g_M,\quad g_M = \max_{x\in[-a_*, b_*]}|g(x)|$$

whence, when $F(p)$ increases (see conditions (1.2) and (1.3)), we have the inequality

$$F(p(x, a))\le F(2g_M/A)$$

Using this inequality and, also, inequalities (2.5) from expression (2.6) and taking account of the inequality $a-\psi(x)\le a_*-a_0$, we obtain the estimate

$$W(x, a)\le F\left(\frac{2g_M}{A}\right)\frac{a_*-a_0}{V_m}\equiv W_M,\quad x, a\in\Pi_* \tag{2.15}$$

We now find the solution $p(x, a)$ of Eq. (2.10) using the method of successive approximations.[6] We just show that the desired solution is the limit of the functional sequence $\{p_k(x, a)\}$ $(k = 1, 2, \ldots)$ of successive approximations

$$p_{k+1}(x, a) = (\mathcal{H}p_k)(x, a), \quad k = 1, 2, \ldots; \quad p_1(x, a) = z(x, a)$$

the existence of which is ensured by Assertion 2 and which, when account is taken of the definition (2.11) of the operator $\mathcal{H}$, can be represented in the form

$$p_k(x, a) = \sum_{i=0}^{k-1} (\mathcal{H}^i z)(x, a), \quad k = 1, 2, \ldots \tag{2.16}$$

$\mathcal{H}$ denotes the i-th action of the operator $\mathcal{H}$. The following properties of the elements of the sequence $\{p_k(x, a)\}$ follow from Assertions 1 and 2, definition (2.11) of the operator $\mathcal{H}$ and by virtue of the continuity on Π_* of the function $z(x, a)$ in the form (2.13):

$$p_k(x, a) \in C(\Pi_*), \quad \int_{-a}^{b} p_k(x, a)dx = Q,$$

$$0 < V_m \le (\mathcal{R}p_k)(a), \quad k = 1, 2, \ldots \tag{2.17}$$

We now take a certain $a_\xi \in (a_0, a_*]$ and define the set

$$\Pi = \{x, a: x \in [-a, b(a)], a \in [a_0, a_\xi]\} \in \Pi_*$$

By a suitable choice of the quantity a_ξ, it is possible to satisfy the inequalities

$$\max_{x, a \in \Pi} F(p_k(x, a)) \le F_+, \quad k = 1, 2, \ldots \tag{2.18}$$

where F_+ is a specified constant. Actually, we make use of the usual norm[6]

$$\|\varphi\| = \max_{x, a \in \Pi} |\varphi(x, a)|, \quad \varphi(x, a) \in C(\Pi)$$

and, on the basis of properties (1.2) and (1.4) of the relation $F(p)$, we write

$$F(\varphi(x, a)) \le F_{1M}\|\varphi(x, a)\|, \quad \varphi(x, a) \in C(\Pi) \tag{2.19}$$

If we now put

$$\varepsilon \equiv \frac{2F_{1M}}{AV_m}(a_\xi - a_0) \in (0, 1) \tag{2.20}$$

then inequality (2.19), when account is taken of definition (2.12) of the operator $\mathcal{H}$, enables us to obtain another inequality, namely, $\|\mathcal{H}_\varphi\| \le \varepsilon\|\varphi\|$ and, using this, from expression (2.16) we obtain the relation

$$\|p_k\| \le \sum_{i=0}^{k-1} \varepsilon^i\|z\| \le \|z\|\sum_{i=0}^{\infty} \varepsilon^i = (1-\varepsilon)^{-1}\|z\|, \quad k = 1, 2, \ldots$$

When account is taken of these relations, substitution of the element $p_k(x, a)$, instead of $\varphi(x, a)$, into inequality (2.19) gives

$$F(p_k(x, a)) \le (1-\varepsilon)^{-1}F_{1M}\|z\|, \quad k = 1, 2, \ldots, \quad x, a \in \Pi \tag{2.21}$$

where, as follows from definition (2.13) of the function $z(x, a)$,

$$\|z\| \le Q/(2l_0) + 2\tilde{g}_M/A; \quad l_0 = (a_0 + b_0)/2, \quad \tilde{g}_M = g_M + 2W_M \tag{2.22}$$

and the quantities g_M and W_M are defined above. Estimate (2.18) is obtained from inequalities (2.21) and (2.22) and, at the same time,

$$F_+ = (1-\varepsilon)^{-1}F_{1M}[Q/(2l_0) + 2\tilde{g}_M/A] \tag{2.23}$$

We will now show that the sequence $\{p_k(x, a)\}$ possesses a fundamental property.[6] Property (1.4) of the function $F(p)$ and, also, the definition (2.2) of the operator $\mathcal{R}$ enable us to write the following inequality for arbitrary elements $p_i(x, a)$ and $p_j(x, a)$ of this sequence:

$$|F(p_j(x, a)) - F(p_i(x, a))| \le F_{1M}\|p_j - p_i\|$$

$$|(\mathcal{R}p_j)(a) - (\mathcal{R}p_i)(a)| \le \gamma F_{1M}\|p_j - p_i\|; \quad \gamma = \max_{a \in [a_0, a_*]} [-1/g'(-a)] > 0$$

Using these inequalities, definition (2.11) of the operator $\mathcal{H}$ as well as estimates (2.17) and (2.18) obtained above, it is possible to arrive at the following result

$$\|(\mathcal{H}p_j) - (\mathcal{H}p_i)\| \le r\|p_j - p_i\|; \quad r = \varepsilon(1 + \gamma F_+ V_m^{-1}) > 0 \tag{2.24}$$

The quantity ε, which is expressed by formula (2.20), is chosen such that $r \in (0, 1)$. In this case, it is known[6] that inequality (2.24) will be a sufficient condition for the sequence $\{p_k(x, a)\}$ to be fundamental in the space $C(\Pi)$. A simple analysis, which takes account of expression (2.23) for F_+, shows that relation $r \in (0, 1)$ holds if $\varepsilon \in (0, \varepsilon_\alpha)$, where

$$\varepsilon_\alpha = 1 + X/2 - \sqrt{(1 + X/2)^2 - 1} \in (0, 1),$$

$$X = \gamma F_{1M} V_m^{-1}[Q/(2l_0) + 2\tilde{g}_M/A]$$

Note that $(0, \varepsilon_\alpha) \subset (0, 1)$ and the inclusion $\varepsilon \in (0, \varepsilon_\alpha)$ therefore ensures that the constraint $\varepsilon \in (0, 1)$ imposed above is satisfied. It follows from definition (2.20) of the quantity ε that the condition $\varepsilon \in (0, \varepsilon_\alpha)$ can be satisfied if the difference $a_\xi - a_0$ is taken to be sufficiently small:

$$a_\xi - a_0 < \varepsilon_\alpha A V_m/(2F_{1M})$$

Consequently, if the step Δ_a is determined from the condition

$$0 < \Delta_a < \varepsilon_\alpha A V_m/(2F_{1M}) \tag{2.25}$$

and we put

$$a_\xi = a_0 + \Delta_a \in (a_0, a_*] \tag{2.26}$$

then the condition $\varepsilon \in (0, \varepsilon_\alpha)$ and, consequently, also the relation $r \in (0, 1)$ will be satisfied and the sequence $\{p_k(x, a)\}$ will be fundamental in $C(\Pi)$.

It is well-known that the space $C(\Pi)$ is complete[7] and the fundamental sequence $\{p_k(x, a)\}$ therefore converges according to the norm to a certain function $p(x, a) \in C(\Pi)$: $\|p - p_k\| \to 0$, $k \to \infty$,[6] where, by virtue of the definition of the norm which is used, this convergence will be pointwise. Taking these facts and expression (2.16) for $p_k(x, a)$ into account, we write

$$\lim_{k \to \infty} p_k(x, a) = \sum_{i=0}^{\infty} (\mathcal{H}^i z)(x, a) = p(x, a) \in C(\Pi) \tag{2.27}$$

By virtue of the second equality of (2.17), the limiting function $p(x, a)$ of the form of (2.27) satisfies the equilibrium condition (1.11) in the interval $[a_0, a_\xi]$. This enables us one to use Assertion 1 and to obtain an inequality analogous to (2.24) for the function $p(x, a)$ which has been found

$$\|(\mathcal{H}p) - (\mathcal{H}p_k)\| \le r\|p - p_k\|$$

which means in the case of condition (2.27) that

$$\|(\mathcal{H}p) - (\mathcal{H}p_k)\| \to 0, \quad k \to \infty$$

This last relation, together with relation (2.27), enables us to write the equalities

$$(\mathcal{H}p)(x,a) = \lim_{k\to\infty}(\mathcal{H}p_k)(x,a) = \lim_{k\to\infty}p_{k+1}(x,a) = p(x,a)$$

indicating that a function $p(x, a)$ of the form of (2.27) satisfies Eq. (2.10) when $x, a \in \Pi$, being its exact solution.

If relations (2.25) and (2.26) allow of the equality $a_\xi = a_*$, then $\Pi = \Pi_*$ and the function $p(x, a)$ obtained using formula (2.27) is a solution of Eq. (2.10) in the whole of the set Π_*. Otherwise, that is, if $a_\xi < a_*$, we put $a_1 = a_\xi$, $\Pi_1 = \Pi|_{a\xi=a1}$ and continue the solution $p(x, a)$ which has been found from the set Π_1 to the set Π_*, taking the dimension a_1 and the shape of the punch

$$\tilde{g}(x) = g(x) + W(x,a_1) - W(0,a_1) \tag{2.28}$$

as the initial values for the process of wear of the punch when $a \geq a_1$. A similar technique is used for the continuation of the solution of differential equations.[8]

It can be verified that all the conditions previously imposed on the function $g(x)$ are satisfied in the case of a punch of the form $\tilde{g}(x)$. Furthermore, the function $\tilde{z}(x, a)$, which is defined in terms of $\tilde{g}(x)$ by equality (2.13), satisfies inequality (2.22) by virtue of the estimate (2.15) for $W(x, a)$ and definition (2.28). All this enables us to repeat the operations of this section, using $\tilde{g}(x)$ as the initial form of the punch. We separately note that, when $g(x)$ is replaced by $\tilde{g}(x)$, the quantities V_m, g_M, W_M and γ do not change according to their definitions and the step Δ_a therefore remains as before (see condition (2.25)). Hence, it is possible to construct a solution $p(x, a)$ of Eq. (2.10) which is continuous in the set

$$\Pi_2 = \{x, a\colon x\in[-a,b],\, a\in[a_1,a_2]\};\quad a_2 = a_1+\Delta_a$$

Using definition (2.28) of the form $\tilde{g}(x)$ of the punch, it is easy to establish that the solutions $p(x, a) \in (\Pi_1)$ and $p(x, a) \in (\Pi_2)$ of Eq. (2.10) which have been found are identical in the line of separation $a = a_1$ of the sets Π_1 and Π_2, constituting a continuous solution of this equation in the set $\Pi_1 \cup \Pi_2$.

If, as before, the quantity $a_2 = a_1 + \Delta_a$ is less than a_*, then, by continuing the solution $p(x, a)$ in the way which has been indicated to values $a \geq a_2$, it is possible, after a finite number of steps Δ_a with respect to a to obtain the required solution $p(x, a) \in C(\Pi_*)$ of Eq. (2.10). This solution, by means of equalities (2.2), (2.6) and (2.8), determines the functions $V(a)$, $W(x, a)$ and $\varphi(a)$ which, together with $p(x, a)$, satisfy Eqs. (1.8)–(1.11) and relation (1.7) on the set Π_*.

If necessary, it is possible to change from the time parameter a to the real time t. For this purpose, it is sufficient, having integrated equality (1.6) with respect to a, to obtain the monotonically increasing relation

$$t(a) = \int_{a_0}^{a}\frac{ds}{V(s)},\quad t(a_0) = 0$$

and to determine the function $a(t)$, which is the inverse of it. When substituted into the expressions which have been found for $b(a)$, $\delta(a)$, $V(a)$, $p(x, a)$, this function gives the relations $b(t)$, $\delta(t)$, $V(t)$, $p(x, t)$, $W(x, t)$. The functions $a(t)$ and $b(t)$ defined in this way satisfy assumptions (1.5) made at the beginning of this section. This follows from relations (1.6), (1.7) and (1.13) and the known properties of an inverse function.[5]

3. The differential equation of wear

The solution $p(x, a) \in C(\Pi_*)$ of the wear-contact problem obtained above can be further investigated using the differential equation which follows from the basic equations of the problem. Actually, in the case of the constraints imposed above on the functions $g(x)$, $F(p)$ and relations (2.1) and (2.5), the derivatives $g'(-a)$, $\partial W(x, a)/\partial a$ are continuous in Π_* and differentiation of the contact condition (2.14) with respect to a, taking account of the wear law (1.8), therefore enables us to arrive at the following differential equation

$$A\frac{\partial p(x,a)}{\partial a} = -g'(-a) - \frac{F(p(x,a))}{V(a)} = \frac{\bar{F}(a) - F(p(x,a))}{V(a)},\quad x,a\in\Pi_* \tag{3.1}$$

where

$$\bar{F}(a) \equiv \frac{1}{2l(a)}\int_{-a}^{b}F(p(x,a))dx \tag{3.2}$$

and the second form of writing the right-hand side of equality (3.1) is obtained using expression (2.2) for $V(a)$.

The initial conditions for (3.1) are specified on the lower boundary of the set Π_*, which is described by the equation $a = \psi(x)$ (see, definition (1.14)). According to equalities (1.10), the contact pressure is equal to zero in the segments $x \in [-a_*, -a_0]$ and $x \in [b_0, b_*]$ of this boundary whereas the initial contact pressure distribution

$$p_0(x) \equiv A^{-1}[g(-a_0) - g(x)] = A^{-1}[g(b_0) - g(x)] \in C[-a_0, b_0]$$

holds in the interval $x \in [-a_0, b_0]$. This distribution is obtained by putting $a = a_0$ and $W(x, a_0) \equiv 0$ in contact condition (2.14). We shall assume that the distribution $p_0(x)$ is non-negative: $p_0(x) \in 0$, $x \in [-a_0, b_0]$ which is ensured by the inequality $g(x) \leq g(-a_0) = g(b_0)$. It can be shown that integral equation (2.10) and deferential equation (3.1) with the specified initial condition are mutually equivalent.

Using (3.1) when $p_0(x) \geq 0$, it is possible to establish the property of the non-negativity of the contact pressure corresponding to its physical meaning

$$p(x,a) \geq 0,\quad x,a\in\Pi_* \tag{3.3}$$

To do this, the following lemma is required.

Lemma 2. *Suppose a function $\varphi(s) \in C[s_1, s_2]$ is given which satisfies the differential equation*

$$\dot{\varphi}(s) \equiv d\varphi(s)/ds = f(s) - \lambda(s)F(\varphi(s)),\quad s\in[s_1,s_2] \tag{3.4}$$

where

$$0 \leq f(s) \in C[s_1,s_2];\quad F(p) = 0,\quad p\leq 0 \tag{3.5}$$

$\lambda(s)$ is an arbitrary function defined in the interval s_1, s_2, s_1 and s_2 are certain specified quantities and $s_1 < s_2$, and that we have $\varphi(s') \geq 0$ for a certain $s' \in [s_1, s_2]$. Then, $\varphi(s) \geq 0$, $s \in [s', s_2]$.

Proof. We will assume that the opposite holds and, in fact, suppose that a $\tilde{s} \in (s', s_2]$ exists such that

$$\varphi(\tilde{s}) < 0 \tag{3.6}$$

When $\varphi(s) \in C(s_1, s_2)$, according to the theorem on the stability of the sign of a continuous function,[5] the existence of a quantity $\rho > 0$, which determines the neighbourhood of the point $\tilde{s}$ where the function $\varphi(s)$ is negative, i.e.,

$$\varphi(s) < 0,\quad s\in(\tilde{s}-\rho,\tilde{s}] \tag{3.7}$$

follows from inequality (3.6).

The values of ρ, which ensure that inequality (3.7) is satisfied, from a non-empty set $\{\rho\}$, with an upper bound $\rho_M = \tilde{s} - s'$, since $\varphi(s) \geq 0$. Consequently, an exact upper bound $\bar{\rho} = \sup\{\rho\}$ exists for this set.[5] By definition, $\bar{\rho} \geq \rho \in \{\rho\}$ and, also, a $\rho' \in \{\rho\}$ is obtained for

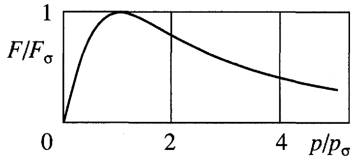

Fig. 2.

any $\varepsilon > 0$ such that $\rho' > \bar{\rho} - \varepsilon$, where, according to inequality (3.7), the last notation means that

$$\varphi(s) < 0, \quad s \in (\tilde{s} - \rho', \tilde{s}] \supset (\tilde{s} - (\bar{\rho} - \varepsilon), \tilde{s}] \equiv (s_0 + \varepsilon, \tilde{s}]; \quad s_0 = \tilde{s} - \bar{\rho}$$

On the basis of these results, the equality $\varphi(s_0) = 0$ can be established and, when this equality is taken into account and also the fact that the magnitude of ε can be as small as desired, we can write

$$\varphi(s) \le 0, \quad s \in [s_0, \tilde{s}]$$

In the case of condition (3.5), this last inequality gives

$$F(\varphi(s)) = 0, \quad s \in [s_0, \tilde{s}]$$

Hence, according to Eq. (3.4), we have

$$\dot{\varphi}(s) = f(s), \quad s \in [s_0, \tilde{s}]$$

The derivative $\dot{\varphi}(s)$ on the left-hand side of the last equality is continuous in the interval $[s_0, \tilde{s}]$ since, according to condition (3.5), the function $f(s)$ possesses the same property. When account is taken of condition (3.5) and the equality $\varphi(s_0) = 0$, which has been established above, we can write

$$0 \le \int_{s_0}^{\tilde{s}} f(s)ds = \int_{s_0}^{\tilde{s}} \dot{\varphi}(s)ds = \varphi(\tilde{s}) - \varphi(s_0) = \varphi(\tilde{s})$$

which contradicts assumption (3.6).

The term $[-g'(-a)]$ which is present in (3.1) is positive, the function $F(p)$ satisfies condition (1.2) and the function $p(x, a)$ takes non-negative values on the lower boundary $a = \psi(c)$ of the set Π_*. All of this enables us to use Lemma 2 and to establish inequalities (3.3).

Equation (3.1) with the second expression for the right-hand side also determines the sign of the derivative $\partial p(x, a)/\partial a$:

$$\operatorname{sign}[\partial p(x, a)/\partial a] = \operatorname{sign}[\bar{F}(a) - F(p(x, a))], \quad x \in [-a, b] \qquad (3.8)$$

The quantity $\bar{F}(a)$ of the form of (3.2) is the mean value of the function $F(p(x, a))$ over the contact area. Relation (3.8) enables us one to carry out a qualitative analysis of the behaviour of the contact pressure during the wear process.

Example. Suppose the function $F(p)$ has the form shown in Fig. 2, where the parameters p_σ, F_σ determine its extremum point. Such a dependence of the rate of wear on the contact pressure holds, for example, in the abrasive wear of polymers.[9] In the case of a weak load, when all the values of $p(x, a)$ are located on the rising part of the $F(p)$ graph, according to condition (3.8) he contact pressure $p(x, a)$ falls (rises) during the course of the wear when $F(p(x, a)) > \bar{F}(a)$ $F(p(x, a)) < \bar{F}(a)$, as a consequence of which the function $F(p(x, a))$ has a tendency to be equalized along the length of the contact ara. Correspondingly, the contact pressure also tends to a constant value $p_s(a)$, where, by virtue of the equilibrium condition (1.11), $p_s(a) = Q/(a + b)$. In the case of heavy loading when, for certain $x \in [-a, b]$, the values of $p(x, a)$ are located on the falling part of the $F(p)$ graph and, for them, $F(p(x, a)) < \bar{F}(a)$, the contact pressure $p(x, a)$ increases during the wear process for these values of x according to equality (3.8), and the values of $F(p(x, a))$ correspondingly decrease, and the latter leads to a further increase in $p(x, a)$ (see, Eq. (3.1)). Hence, in the case of a heavy load, an unbounded increase in the contact pressure is possible with time in the most loaded part of the contact area. In the remaining part of the contact area, the function $p(x, a)$ will, as before, tend to adopt a constant value. Note that, in the case of a linear wear law, equalization of the contact pressure curve with time is well known.[2]

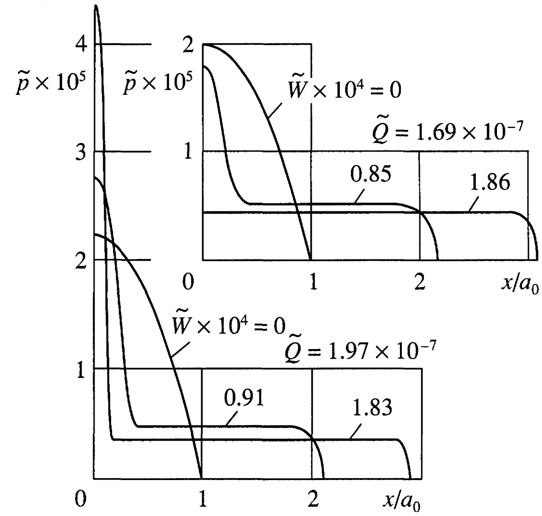

Fig. 3.

In order to check the results of the above qualitative analysis, a numerical solution of Eq. (3.1) was constructed for the case of a symmetric parabolic punch $g(x) = x_2/(2R)$. The quantity

$$\tilde{W}_M(a) = \max_{x \in [-a, a]} W(x, a)/R$$

which characterizes the maximum wear over the contact area, was used to describe the extent of the wear. The calculated dimensional contact pressure curves $\tilde{p} = AR^{-1}p$, corresponding to different values of $\tilde{W}_M$ for two different loads $\tilde{Q} = AR^{-2}Q$, are shown in Fig. 3. The data presented are in agreement with the results of the qualitative analysis of the behaviour of the contact pressure during the course of the wear.

4. Conclusions

1. Integral equation (2.10) and differential equation (1.3) of the wear contact problem for a Winkler foundation have been obtained in the case of a non-linear wear law.
2. A successive approximation procedure has been proposed which enables us to find the exact solution of the integral equation in the space of continuous functions. To do this, a lower estimate was obtained for the rate of increase in the dimension of the contact area.
3. The property of the non-negativity of the contact pressure when its initial values are non-negative has been established on the basis of the differential equation.
4. It has been shown by qualitative analysis and calculations that the character of the non-linearity of the wear law can have a substantial effect on the behaviour of the contact pressure during the course of the wear.

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References

1. Gorbunov-Posadov MI, Malikova TA. *The Design of Structures on an Elastic Foundation*. Moscow: Stroiizdat; 1973.
2. Vorovich II, Aleksandrov VM, editors. *The Mechanics of Contact Interactions*. Moscow: Fizmatlit; 2001.
3. Levina ZM, Reshetov DN. *The Contact Stiffness of Machines*. Moscow: Mashinostroyeniye; 1971.
4. Goryacheva IG, Dobychin MN. *Contact Problems in Tribology*. Moscow: Mashinostroyeniye; 1988.
5. Il'in VA, Poznyak EG. *Principles of Mathematical Analysis*, Part 1. Moscow: Nauka; 1971. Part 2; 1973.
6. Kolmogorov AN, Fomin SV. *Elements of the Theory of Functions and Functional Analysis*. New York: Dover; 1994.
7. Vorovich II, Lebedev LP. *Functional Analysis and its Application in Continuum Mechanics*. Moscow: Vuz. Kn; 2000.
8. Stepanov VV. *A Course of Differential Equations*. Moscow: Gostekhizdat; 1953.
9. Tenenbaum MM. *Resistance to Abrasive Wear*. Moscow: Mashinostroyeniye; 1976.

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